JOURNAL OF FUNCTIONAL ANALYSIS 35, 230–250 (1980)

Unbounded Generalizations of Left Hilbert Algebras, II

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Communicated by the Editors

Received November 17, 1978

The first purpose of this paper is to study the classification of unbounded left Hilbert algebras. The second purpose is to investigate the unbounded left Hilbert algebras generated by positive linear functionals on a *-algebra. The final purpose is to study the classification of positive linear functionals.

1. INTRODUCTION

In this paper we continue our study of unbounded left Hilbert algebras begun in a previous paper [8]. In particular, we investigate the relation between positive linear functionals on a *-algebra and unbounded left Hilbert algebras.

Let f be a positive linear functional on a *-algebra $\mathbf{A}$. The elements a of $\mathbf{A}$ such that $f(a^*a) = 0$ form a left ideal $\mathbf{N}_f$ in $\mathbf{A}$. If $a \in \mathbf{A}$ we denote by $\lambda_f(a)$ the coset of $\mathbf{A}/\mathbf{N}_f \equiv \lambda_f(\mathbf{A})$ which contains a and we define an inner product by $(\lambda_f(a) \mid \lambda_f(b)) = f(b^*a)$. We consider under what conditions $\lambda_f(\mathbf{A})$ becomes an unbounded left Hilbert algebra.

In [7] we showed that if f is abelian, then $\lambda_f(\mathbf{A})$ is an unbounded Hilbert algebra [4], and using the classification of unbounded Hilbert algebras [5], we assigned classifications to abelian positive linear functionals. The primary purpose of this paper is to consider this problem for nonabelian positive linear functionals.

In Section 2 we study the classification of unbounded left Hilbert algebras.

In Section 3 we define the notation of quasi-abelian positive linear functionals and show that if f is quasi-abelian, then $\lambda_f(\mathbf{A})$ is an unbounded left Hilbert algebra.

In Section 4 we study positive linear functionals g with $g \leqslant f$ (where f is a quasi-abelian positive linear functional).

In Section 5, using the classification of unbounded left Hilbert algebras, we give the classifications of quasi-abelian positive linear functionals.

0022-1236/80/020230–21$02.00/0

2. Classification of Unbounded Left Hilbert Algebras

We begin with some basic terminology.

If A and B are linear operators in a Hilbert space $\mathfrak{H}$ with domains $\mathscr{D}(A)$ and $\mathscr{D}(B)$, then we say A is an extension of B, denoted by $A \supset B$, if $\mathscr{D}(A) \supset \mathscr{D}(B)$ and $A\xi = B\xi$ for all $\xi \in \mathscr{D}(B)$. If A is a closable operator, then we denote by $\bar{A}$ the smallest closed extension of A. Let $\mathscr{A}$ be a set of closable operators in $\mathfrak{H}$. Then, we denote by $\bar{\mathscr{A}}$ the set $\{\bar{A}; A \in \mathscr{A}\}$. If A is a linear operator with dense domain $\mathscr{D}(A)$, then we denote by A^* the hermitian adjoint of A. Let A and B be closed operators in $\mathfrak{H}$. If $A + B$ is closable, then $\overline{A + B}$ is called the strong sum of A and B and is denoted by $A \boldsymbol{+} B$. The strong product is likewise defined to be $\overline{AB}$ if it exists and is denoted by $A \cdot B$. The strong scalar multiplication of $\lambda \in \mathbb{C}$ (the field of complex numbers) and A is defined by $\lambda \cdot A = \lambda A$ if $\lambda \neq 0$ and $\lambda \cdot A = 0$ if $\lambda = 0$.

Let $\mathfrak{A}_0$ be a left Hilbert algebra with involution $\#$ in a Hilbert space $\mathfrak{H}$ and $\mathscr{U}_0(\mathfrak{A}_0)$ (resp. $\mathscr{V}_0(\mathfrak{A}_0)$) be the left (resp. right) von Neumann algebra of $\mathfrak{A}_0$. Let $F_{\mathfrak{A}_0} : \eta \to \eta^\flat$ be the adjoint involution of the involution $\xi \in \mathfrak{A}_0 \to \xi^\# \in \mathfrak{A}_0$ and $\mathscr{D}^\flat(\mathfrak{A}_0)$ the definition domain of $F_{\mathfrak{A}_0}$. Take and fix an element η of $\mathscr{D}^\flat(\mathfrak{A}_0)$. Define an operator $\pi_0'(\eta)$ by

$$\pi_0'(\eta)\xi = \overline{\pi_0(\xi)}\eta, \qquad \xi \in \mathfrak{A}_0,$$

where π_0 denotes the left regular representation of $\mathfrak{A}_0$. Then, $\pi_0'(\eta)^* \supset \overline{\pi_0'(\eta^\flat)}$ and $\overline{\pi_0'(\eta)}$ is affiliated with $\mathscr{V}_0(\mathfrak{A}_0)$ (is denoted by $\overline{\pi_0'(\eta)}\ \eta \mathscr{V}_0(\mathfrak{A}_0)$). We define

$$\mathfrak{A}_0' = \{\eta \in \mathscr{D}^\flat(\mathfrak{A}_0); \overline{\pi_0'(\eta)} \in \mathscr{B}(\mathfrak{H})\},$$

where $\mathscr{B}(\mathfrak{H})$ denotes the set of all bounded linear operators on $\mathfrak{H}$. Then, $\mathfrak{A}_0'$ is an involutive algebra with involution $\eta \to \eta^\flat$ and π_0' is an antirepresentation of $\mathfrak{A}_0'$ in $\mathfrak{H}$. Let $\mathscr{D}^\#(\mathfrak{A}_0)$ be the definition domain of the adjoint involution $S_{\mathfrak{A}_0}$ of the involution $\eta \to \eta^\flat$. For each $\xi \in \mathscr{D}^\#(\mathfrak{A}_0)$, define an operator $\pi_0(\xi)$ by

$$\pi_0(\xi)\eta = \overline{\pi_0'(\eta)}\xi, \qquad \eta \in \mathfrak{A}_0'.$$

Then, $\pi_0(\xi)^* \supset \overline{\pi_0(\xi^\#)}$ and $\overline{\pi_0(\xi)}\ \eta \mathscr{U}_0(\mathfrak{A}_0)$. We define

$$\mathfrak{A}_0'' = \{\xi \in \mathscr{D}^\#(\mathfrak{A}_0); \overline{\pi_0(\xi)} \in \mathscr{B}(\mathfrak{H})\}.$$

$\mathfrak{A}_0''$ is an involutive algebra with involution $\xi \to \xi^\#$, which contains $\mathfrak{A}_0$ as an involutive subalgebra. The definitions and notations of Takesaki's paper [10] are usually used below without reference.

We now define an unbounded left Hilbert algebra which is an unbounded generalization of a left Hilbert algebra as follows:

Let $\mathfrak{A}$ be a pre-Hilbert space with inner product $(\ |\ )$ and an algebra with involution $\#$. Let $\mathfrak{H}$ be the completion of the pre-Hilbert space $\mathfrak{A}$. Suppose that

$$\text{(i)}\quad (\xi\eta \mid \zeta) = (\eta \mid \xi^{\#}\zeta), \qquad \xi, \eta, \zeta \in \mathfrak{A}.$$

Then, we set

$$\pi(\xi)\,\eta = \xi\eta, \qquad \xi, \eta \in \mathfrak{A}.$$

By (i), $\pi(\xi)$ is a closable operator in $\mathfrak{H}$ with domain $\mathfrak{A}$ and $\pi(\xi)^* \supset \overline{\pi(\xi^{\#})}$. We set

$$\mathfrak{A}_0 = \{\xi \in \mathfrak{A}; \overline{\pi(\xi)} \in \mathscr{B}(\mathfrak{H})\}.$$

If $\mathfrak{A}$ satisfies condition (i) above and the following conditions:

(ii) $\mathfrak{A}_0$ is a left Hilbert algebra with the involution $\#$,

(iii) the involution $S_{\mathfrak{A}_0}$ is an extension of the involution $\#$ on $\mathfrak{A}$,

(iv) $\overline{\pi_0(\xi)} \subset \overline{\pi(\xi)}$ for every $\xi \in \mathfrak{A}$ (where π_0 denotes the left regular representation of $\mathfrak{A}_0$),

then $\mathfrak{A}$ is called an unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$ and π is called the left regular representation of $\mathfrak{A}$. An unbounded left Hilbert algebra $\mathfrak{A}$ over $\mathfrak{A}_0$ is called purely unbounded if $\mathfrak{A} \neq \mathfrak{A}_0$. If $\mathfrak{A}_0$ is an acheved left Hilbert algebra, then $\mathfrak{A}$ is said to be achieved.

For details, the reader is referred to [8].

In this section we give the classification of unbounded left Hilbert algebras.

We have the following

LEMMA 2.1. *If $\mathfrak{A}_0$ is a left Hilbert algebra in a Hilbert space $\mathfrak{H}$ and E is a projection in $\mathscr{U}_0(\mathfrak{A}_0) \cap \mathscr{V}_0(\mathfrak{A}_0)$, then*

(i) *$E\mathfrak{A}_0$ is a left Hilbert algebra in $E\mathfrak{H}$ under the operations: $(E\xi)(E\eta) = E\xi\eta$, $(E\xi)^{\#} = E\xi^{\#}$;*

(ii) *$\mathscr{D}^{\flat}(E\mathfrak{A}_0) = E\mathscr{D}^{\flat}(\mathfrak{A}_0)$ and for each $\eta \in \mathscr{D}^{\flat}(\mathfrak{A}_0)$*

$$(E\eta)^{\flat} = E\eta^{\flat}, \qquad \overline{\pi_0'(E\eta)} = \overline{\pi_0'(\eta)}/E\mathfrak{H},$$

where $\overline{\pi_0'(\eta)}/E\mathfrak{H}$ denotes the restriction of $\overline{\pi_0'(\eta)}$ to $E\mathfrak{H}$;

(iii) *$(E\mathfrak{A}_0)' = E\mathfrak{A}_0'$;*

(iv) *$\mathscr{D}^{\#}(E\mathfrak{A}_0) = E\mathscr{D}^{\#}(\mathfrak{A}_0)$ and for each $\xi \in \mathscr{D}^{\#}(\mathfrak{A}_0)$*

$$(E\xi)^{\#} = E\xi^{\#}, \qquad \overline{\pi_0(E\xi)} = \overline{\pi_0(\xi)}/E\mathfrak{H};$$

(v) *$(E\mathfrak{A}_0)'' = E\mathfrak{A}_0''$.*

THEOREM 2.1. *If $\mathfrak{A}$ is an unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$ and E is a projection in $\mathscr{U}_0(\mathfrak{A}_0) \cap \mathscr{V}_0(\mathfrak{A}_0)$, then $E\mathfrak{A}$ is an unbounded left Hilbert algebra over $E\mathfrak{A} \cap E\mathfrak{A}_0''$ in $E\mathfrak{H}$ under the operations: $(E\xi)(E\eta) = \pi(E\xi)(E\eta) = E\xi\eta$, $(E\xi)^{\#} = E\xi^{\#}$.*

Proof. From Lemma 3.1 in [8], $\overline{\pi(\xi)}\ \eta\mathscr{U}_0(\mathfrak{A}_0)$ for each $\xi \in \mathfrak{A}$, and so it follows that $((E\xi)(E\eta) \mid E\zeta) = (E\eta \mid (E\xi)^{\#}(E\zeta))$ for each $\xi, \eta, \zeta \in \mathfrak{A}$. It follows from Lemma 2.1(iv) that

(a) $\mathscr{D}^{\#}(E\mathfrak{A}_0) = E\mathscr{D}^{\#}(\mathfrak{A}_0) \supset E\mathfrak{A}$.

Further, we have

(b) $\overline{\pi(E\xi)} \supset \overline{\pi_0(E\xi)}$ for every $\xi \in \mathfrak{A}$,

where π_0 denotes the left regular representation of the left Hilbert algebra $E\mathfrak{A}_0$. In fact, since $\overline{\pi(\xi)} \supset \overline{\pi_0(\xi)}$, $\eta \in \mathscr{D}(\overline{\pi(\xi)})$ for every $\eta \in \mathfrak{A}_0'$, and so there exists a sequence $\{\eta_n\}$ in $\mathfrak{A}$ such that $\lim_{n\to\infty} \eta_n = \eta$ and $\lim_{n\to\infty} \pi(\xi)\,\eta_n = \overline{\pi(\xi)}\,\eta$. Then we have

$$\lim_{n\to\infty} E\eta_n = E\eta,$$

$$\lim_{n\to\infty} \pi(E\xi)\, E\eta_n = \lim_{n\to\infty} E\pi(\xi)\,\eta_n = E\overline{\pi(\xi)}\eta.$$

Hence, $E\eta \in \mathscr{D}(\overline{\pi(E\xi)})$ and

$$\begin{aligned} \overline{\pi(E\xi)}\, E\eta &= E\overline{\pi(\xi)}\eta = E\pi_0'(\eta)\xi \\ &= \pi_0'(\eta)\, E\xi = \pi_0'(E\eta)\, E\xi \\ &= \pi_0(E\xi)\, E\eta. \end{aligned}$$

Using this fact and Lemma 2.1(iv), (v), we can easily prove that

$$\begin{aligned} (E\mathfrak{A})_0 &\equiv \{E\xi \in E\mathfrak{A};\ \overline{\pi(E\xi)} \subset \mathscr{B}(E\mathfrak{H})\} \\ &= E\mathfrak{A} \cap E\mathfrak{A}_0''. \end{aligned}$$

Hence, $(E\mathfrak{A})_0$ becomes a left Hilbert algebra equivalent to the left Hilbert algebra $E\mathfrak{A}_0$. Consequently it follows from (a) and (b) that $E\mathfrak{A}$ is an unbounded left Hilbert algebra over $E\mathfrak{A} \cap E\mathfrak{A}_0''$ in $E\mathfrak{H}$.

Let $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of unbounded left Hilbert algebras $\mathfrak{A}_\lambda$ over $(\mathfrak{A}_\lambda)_0$ in $\mathfrak{H}_\lambda$. Let $\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda$ be the set of all elements of the Cartesian product $\mathsf{X}_{\lambda\in\Lambda} \mathfrak{A}_\lambda$ with only a finite number of nonzero coordinates. In particular, if Λ is a finite set $\{1, 2,..., n\}$, then $\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda$ is also denoted by $\mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$. Define the operations and inner product on $\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda$ as follows:

$$\{\xi_\lambda\}\{\eta_\lambda\} = \pi(\{\xi_\lambda\})\{\eta_\lambda\} = \{\xi_\lambda\eta_\lambda\},$$

$$\{\xi_\lambda\}^{\#} = \{\xi_\lambda{}^{\#}\},$$

$$(\{\xi_\lambda\} \mid \{\eta_\lambda\}) = \sum_{\lambda\in\Lambda} (\xi_\lambda \mid \eta_\lambda).$$

Then, we show that $\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda$ is an unbounded left Hilbert algebra over $\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0$ in the direct sum $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ of the Hilbert spaces $\mathfrak{H}_\lambda$.

LEMMA 2.2. *$\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0$ is a left Hilbert algebra in $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ and it satisfies:*

(i) $(\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0)' = \{\{\eta_\lambda\} \in \bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda ;\ \eta_\lambda \in (\mathfrak{A}_\lambda)_0'$ *for all* $\lambda \in \Lambda$ *and* $\sum_{\lambda\in\Lambda} \| \eta_\lambda{}^\flat \|^2 < \infty\}$;

(ii) $\{\eta_\lambda\}^\flat = \{\eta_\lambda{}^\flat\}$ *for each* $\{\eta_\lambda\} \in (\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0)'$;

(iii) $(\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0)'' = \{\{\xi_\lambda\} \in \bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda ;\ \xi_\lambda \in (\mathfrak{A}_\lambda)_0''$ *for all* $\lambda \in \Lambda$ *and* $\sum_{\lambda\in\Lambda} \| \xi_\lambda{}^{\#} \|^2 < \infty\}$;

(iv) $\{\xi_\lambda\}^{\#} = \{\xi_\lambda{}^{\#}\}$ *for each* $\{\xi_\lambda\} \in (\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0)''$.

Proof. From Theorem 11.2 in [10] it follows that $\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0$ is a left Hilbert algebra in $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$. Assertions (i) through (iv) are easily proved.

Let A_λ be a linear operator in $\mathfrak{H}_\lambda$ with dense domain $\mathscr{D}(A_\lambda)$. Then, we define a linear operator $\{A_\lambda\}$ in $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ with domain $\mathscr{D}(\{A_\lambda\})$ as follows:

$$\mathscr{D}(\{A_\lambda\}) = \left\{\{x_\lambda\} \in \bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda ;\ x_\lambda \in \mathscr{D}(A_\lambda) \text{ for all } \lambda \in \Lambda \text{ and } \sum_{\lambda\in\Lambda} \| A_\lambda x_\lambda \|^2 < \infty\right\},$$

$$\{A_\lambda\}\{x_\lambda\} = \{A_\lambda x_\lambda\}, \qquad \{x_\lambda\} \in \mathscr{D}(\{A_\lambda\}).$$

Then we have

LEMMA 2.3. (i) *If A_λ is a closed operator for all $\lambda \in \Lambda$, then $\{A_\lambda\}$ is a closed operator;*

(ii) *If A_λ is a closable operator for all $\lambda \in \Lambda$, then $\{A_\lambda\}$ is a closable operator such that $\overline{\{A_\lambda\}} = \{\overline{A_\lambda}\}$ and $\{A_\lambda\}^* = \{A_\lambda^*\}$.*

THEOREM 2.1. *$\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda$ is an unbounded left Hilbert algebra over $\sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_0$ in $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$.*

Proof. It is easily proved that $(\{\xi_\lambda\}\{\eta_\lambda\} \mid \{\zeta_\lambda\}) = (\{\eta_\lambda\} \mid \{\xi_\lambda\}^{\#}\{\zeta_\lambda\})$ for each $\{\xi_\lambda\}$, $\{\eta_\lambda\}$, $\{\zeta_\lambda\} \in \sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda$ and $(\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda)_0 = \{\{\xi_\lambda\} \in \sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda ;\ \overline{\pi(\{\xi_\lambda\})} \in$

$\mathscr{B}(\bigoplus_{\lambda\in\Lambda}\mathfrak{H}_\lambda)\} = \sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_0$. By Lemma 2.2, $(\sum^{\oplus}_{\lambda\in\Lambda}\mathfrak{A}_\lambda)_0$ is a left Hilbert algebra in $\bigoplus_{\lambda\in\Lambda}\mathfrak{H}_\lambda$.

We next show that $\mathscr{D}^{\#}(\sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_0)\supset\sum^{\oplus}_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. For each $\{\xi_\lambda\}\in\sum^{\oplus}_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ and $\{\eta_\lambda\}\in(\sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_0)'$ it follows from Lemma 2.2(i), (ii) that

$$\begin{aligned}(\{\xi_\lambda\}\mid\{\eta_\lambda\}) &= \sum_{\lambda\in\Lambda}(\xi_\lambda\mid\eta_\lambda) = \sum_{\lambda\in\Lambda}(\eta_\lambda{}^\flat\mid\xi_\lambda{}^\#)\\ &= (\{\eta_\lambda{}^\flat\}\mid\{\xi_\lambda{}^\#\}) = (\{\eta_\lambda\}^\flat\mid\{\xi_\lambda\}^\#).\end{aligned}$$

Hence,

$$\{\xi_\lambda\}\in\mathscr{D}^{\#}\left(\sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_0\right)\qquad\text{and}\qquad S_{\sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_0}\{\xi_\lambda\} = \{\xi_\lambda\}^\#.$$

Finally, we show that $\overline{\pi(\{\xi_\lambda\})}\supset\overline{\pi_0(\{\xi_\lambda\})}$ for each $\{\xi_\lambda\}\in\sum^{\oplus}_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. From Lemma 2.3 it follows immediately that

$$\overline{\pi(\{\xi_\lambda\})} = \overline{\{\pi(\xi_\lambda)\}} = \{\overline{\pi(\xi_\lambda)}\}$$

$$\overline{\pi_0(\{\xi_\lambda\})} = \overline{\{\pi_0(\xi_\lambda)\}} = \{\overline{\pi_0(\xi_\lambda)}\}.$$

This implies that

$$\overline{\pi(\{\xi_\lambda\})} = \{\overline{\pi(\xi_\lambda)}\}\supset\{\overline{\pi_0(\xi_\lambda)}\} = \overline{\pi_0(\{\xi_\lambda\})}.$$

This completes the proof.

Definition 2.1. $\sum^{\oplus}_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ is called the direct sum of the unbounded left Hilbert algebras $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$.

Definition 2.2. Let $\mathfrak{A}$ be an unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$. If there exists a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal projections E_λ in $\mathscr{U}_0(\mathfrak{A}_0)\cap\mathscr{V}_0(\mathfrak{A}_0)$ such that $\sum_{\lambda\in\Lambda}E_\lambda = I$ and $E_\lambda\mathfrak{A}$ is a left Hilbert algebra for every $\lambda\in\Lambda$, then $\mathfrak{A}$ is called weakly unbounded. If there does not exist any nonzero projection E in $\mathscr{U}_0(\mathfrak{A}_0)\cap\mathscr{V}_0(\mathfrak{A}_0)$ such that $E\mathfrak{A}$ is a left Hilbert algebra, then $\mathfrak{A}$ is called strictly unbounded.

If $\mathfrak{A}$ is a weakly unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$, then $\mathfrak{A}$ is regarded as an unbounded left Hilbert algebra in $\bigoplus_{\lambda\in\Lambda}E_\lambda\mathfrak{H}$ by the map: $\xi\in\mathfrak{A}\to\{E_\lambda\xi\}\in\bigoplus_{\lambda\in\Lambda}E_\lambda\mathfrak{H}$. For each $\lambda\in\Lambda$ we have $E_\lambda\mathfrak{A}_0\subset E_\lambda\mathfrak{A}\subset E_\lambda\mathfrak{A}_0''$. In particular, if $\mathfrak{A}$ is achieved, then $E_\lambda\mathfrak{A} = E_\lambda\mathfrak{A}_0$ for every $\lambda\in\Lambda$ and $\mathfrak{A}$ is regarded as an unbounded left Hilbert algebra in $\bigoplus_{\lambda\in\Lambda}E_\lambda\mathfrak{H}$ containing the left Hilbert algebra $\sum^{\oplus}_{\lambda\in\Lambda}E_\lambda\mathfrak{A}_0{}^2$.

THEOREM 2.3. *If $\mathfrak{A}$ is an unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$, then there exists a projection E in $\mathscr{U}_0(\mathfrak{A}_0) \cap \mathscr{V}_0(\mathfrak{A}_0)$ such that*

(i) *$E\mathfrak{A}$ is weakly unbounded;*

(ii) *$(I - E)\mathfrak{A}$ is strictly unbounded;*

(iii) *$\mathfrak{A}$ is a *-subalgebra of the unbounded left Hilbert algebra $E\mathfrak{A} \oplus (I - E)\mathfrak{A}$;*

(iv) *the left Hilbert algebra $\mathfrak{A}_0$ is equivalent to the left Hilbert algebra $(E\mathfrak{A} \oplus (I - E)\mathfrak{A})_0$.*

Proof. Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be a maximal family of mutually orthogonal projections in $\mathscr{U}_0(\mathfrak{A}_0) \cap \mathscr{V}_0(\mathfrak{A}_0)$ such that $E_\lambda\mathfrak{A}$ is a left Hilbert algebra. We set $E = \sum_{\lambda\in\Lambda} E_\lambda$. Then, it is easily proved that E satisfies the assertions of the theorem.

Let $\mathfrak{A}$ be an unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$. It follows from Lemma 3.1 in [8] that $\overline{\pi(\xi)}$ is affiliated with $\mathscr{U}_0(\mathfrak{A}_0)$. We set

$$\mathscr{D}_{\mathfrak{A}} = \bigcap_{\xi\in\mathscr{D}} \mathscr{D}(\overline{\pi(\xi)}),$$

$$\tilde{\pi}(\xi) = \overline{\pi(\xi)}/\mathscr{D}_{\mathfrak{A}}.$$

Then, $\tilde{\pi}(\mathfrak{A}) \equiv \{\tilde{\pi}(\xi);\ \xi \in \mathfrak{A}\}$ is a closed $\#$-algebra on $\mathscr{D}_{\mathfrak{A}}$ (cf. [3]), which is called the closed left $\#$-algebra of $\mathfrak{A}$. If a *-algebra generated by $\overline{\pi(\mathfrak{A})}$ and $\mathscr{U}_0(\mathfrak{A}_0)$ under the operations of strong sum, strong product, strong scalar multiplication, and adjoint is defined, then it becomes an EW^*-algebra over $\mathscr{U}_0(\mathfrak{A}_0)$ (cf. [2]), which is called the left EW^*-algebra of $\mathfrak{A}$ and is denoted by $\overline{\mathscr{U}}(\mathfrak{A})$. Further, if $\overline{\mathscr{U}}(\mathfrak{A})$ has a common dense domain, i.e., $\mathscr{D}(\overline{\mathscr{U}}(\mathfrak{A})) = \bigcap_{A\in\overline{\mathscr{U}}(\mathfrak{A})} \mathscr{D}(A)$ is dense in $\mathfrak{H}$, then $\overline{\mathscr{U}}(\mathfrak{A})/\mathscr{D}(\overline{\mathscr{U}}(\mathfrak{A}))$ is a closed $EW^\#$-algebra over $\mathscr{U}_0(\mathfrak{A}_0)$ (cf. [3]), which is called the closed left $EW^\#$-algebra of $\mathfrak{A}$ and is denoted by $\tilde{\mathscr{U}}(\mathfrak{A})$. If the closed left $EW^\#$-algebra $\tilde{\mathscr{U}}(\mathfrak{A})$ of $\mathfrak{A}$ is defined, then $\mathfrak{A}$ is called representable.

For a more complete discussion concerning unbounded left Hilbert algebras the reader is referred to [8].

Let $\{\mathscr{M}_\lambda\}_{\lambda\in\Lambda}$ be a family of *-algebras of bounded operators $\mathscr{M}_\lambda$ on Hilbert spaces $\mathfrak{H}_\lambda$. We denote by $\prod_{\lambda\in\Lambda} \mathscr{M}_\lambda$ the set $\{\{A_\lambda\};\ A_\lambda \in \mathscr{M}_\lambda\}$ of closed operators $\{A_\lambda\}$ in $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$. Then, $\prod_{\lambda\in\Lambda} \mathscr{M}_\lambda$ is a *-algebra of closed operators in $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ under the operations of strong sum, strong product, strong scalar multiplication, and adjoint. In particular, if $\mathscr{M}_\lambda$ is a von Neumann algebra for every $\lambda \in \Lambda$, then $\prod_{\lambda\in\Lambda} \mathscr{M}_\lambda$ is an EW^*-algebra over the direct sum $\bigoplus_{\lambda\in\Lambda} \mathscr{M}_\lambda$ of the von Neumann algebras $\mathscr{M}_\lambda$.

DEFINITION 2.3. An $EW^\#$-algebra $\mathscr{M}$ is called weakly unbounded if there exists a family $\{\mathscr{M}_\lambda\}_{\lambda\in\Lambda}$ of von Neumann algebras such that $\overline{\mathscr{M}}$ is a *-subalgebra of the EW^*-algebra $\prod_{\lambda\in\Lambda} \mathscr{M}_\lambda$ and $\overline{\mathscr{M}}_b = \bigoplus_{\lambda\in\Lambda} \mathscr{M}_\lambda$.

LEMMA 2.4. *If $\mathfrak{A}$ is a weakly unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$, that is, there exists a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal projections E_λ in $\mathscr{U}_0(\mathfrak{A}_0) \cap \mathscr{V}_0(\mathfrak{A}_0)$ such that $\sum_{\lambda\in\Lambda} E_\lambda = I$ and $E_\lambda\mathfrak{A}$ is a left Hilbert algebra for every $\lambda \in \Lambda$, then $\overline{\pi(\xi)} = \overline{\{\pi_0(E_\lambda\xi)\}}$ for every $\xi \in \mathfrak{A}$.*

Proof. For each $\eta \in \mathfrak{A}$ we have

$$\sum_{\lambda\in\Lambda} \| \overline{\pi_0(E_\lambda\xi)}\, E_\lambda\eta \|^2 = \sum_{\lambda\in\Lambda} \| E_\lambda\xi\eta \|^2 = \| \xi\eta \|^2.$$

Hence, $\eta = \{E_\lambda\eta\} \in \mathscr{D}(\{\overline{\pi_0(E_\lambda\xi)}\})$ and $\pi(\xi)\,\eta = \overline{\{\pi_0(E_\lambda\xi)\}}\{E_\lambda\eta\}$. Thus, $\overline{\pi(\xi)} \subset \overline{\{\pi_0(E_\lambda\xi)\}}$. Since $\overline{\pi_0(E_\lambda\xi)} = \overline{\pi(\xi)}\, E_\lambda$, for each $x = \{x_\lambda\} \in \mathscr{D}(\{\overline{\pi_0(E_\lambda\xi)}\})$ we have that $x_\lambda \in \mathscr{D}(\overline{\pi(\xi)})$ for each $\lambda \in \Lambda$ and $\sum_{\lambda\in\Lambda} \| \overline{\pi(\xi)}\, x_\lambda \|^2 = \sum_{\lambda\in\Lambda} \| \overline{\pi_0(E_\lambda\xi)}\, x_\lambda \|^2 < \infty$. Hence, $x \in \mathscr{D}(\overline{\pi(\xi)})$.

THEOREM 2.4. *If $\mathfrak{A}$ is a weakly unbounded left Hilbert algebra over $\mathfrak{A}_0$ in $\mathfrak{H}$, then it is representable. Further, the closed left $EW^{\#}$-algebra $\tilde{\mathscr{U}}(\mathfrak{A})$ is weakly unbounded.*

Proof. Since $\mathfrak{A}$ is weakly unbounded, there is a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal projections in $\mathscr{U}_0(\mathfrak{A}) \cap \mathscr{V}_0(\mathfrak{A}_0)$ such that $\sum_{\lambda\in\Lambda} E_\lambda = I$ and $E_\lambda\mathfrak{A}$ is a left Hilbert algebra for every $\lambda \in \Lambda$. From Lemma 2.4 it follows that $\overline{\pi(\xi)} \in \prod_{\lambda\in\Lambda} \mathscr{U}_0(E_\lambda\mathfrak{A}_0)$ for every $\xi \in \mathfrak{A}$. Also, $\mathscr{U}_0(\mathfrak{A}_0) = \bigoplus_{\lambda\in\Lambda} \mathscr{U}_0(E_\lambda\mathfrak{A}_0)$. Hence, a *-algebra $\overline{\mathscr{U}}(\mathfrak{A})$ generated by $\overline{\pi(\mathfrak{A})}$ and $\mathscr{U}_0(\mathfrak{A}_0)$ is a *-subalgebra of the EW^*-algebra $\prod_{\lambda\in\Lambda} \mathscr{U}_0(E_\lambda\mathfrak{A}_0)$ such that $\overline{\mathscr{U}}(\mathfrak{A})_b = \mathscr{U}_0(\mathfrak{A}_0)$ and $\mathscr{D}(\overline{\mathscr{U}}(\mathfrak{A})) = \bigcap_{X\in\overline{\mathscr{U}}(\mathfrak{A})} \mathscr{D}(X) \supset \sum^{\oplus}_{\lambda\in\Lambda} E_\lambda\mathfrak{H}$. This implies that the closed left $EW^{\#}$-algebra $\tilde{\mathscr{U}}(\mathfrak{A})$ of $\mathfrak{A}$ is well defined and it is wekly unbounded.

3. QUASI-ABELIAN POSITIVE LINEAR FUNCTIONALS

Let f be a positive linear functional on a *-algebra $\mathbf{A}$. The elements a in $\mathbf{A}$ such that $f(a^*a) = 0$ form a left ideal $\mathbf{N}_f$ in $\mathbf{A}$. If $a \in \mathbf{A}$, we denote by $\lambda_f(a)$ the coset of $\mathbf{A}/\mathbf{N}_f \equiv \lambda_f(\mathbf{A})$ which contains a and we define an inner product by: $(\lambda_f(a) \mid \lambda_f(b)) = f(b^*a)$. Then, $\lambda_f(\mathbf{A})$ becomes a pre-Hilbert space. Let $\mathfrak{H}_f$ be the completion of $\lambda_f(\mathbf{A})$. If $a \in \mathbf{A}$, we denote by $\pi_f(a)$ the operator in $\mathfrak{H}_f$ whose domain is $\lambda_f(\mathbf{A})$ and which maps $\lambda_f(b)$ into $\lambda_f(ab)$. If the condition: $f(a^*a) = 0$ implies $a = 0$, then f is called faithful.

Throughout this section let $\mathbf{A}$ be a *-algebra with identity e and f a positive linear functional on $\mathbf{A}$. We set

$$\mathbf{A}_b(f) = \{a \in \mathbf{A};\, \overline{\pi_f(a)} \in \mathscr{B}(\mathfrak{H}_f)\}.$$

Then, $\mathbf{A}_b(f)$ is a *-subalgebra of $\mathbf{A}$.

In this section we define the notation of quasi-abelian positive linear functionals and show that if f is quasi-abelian, then $\lambda_f(\mathbf{A})$ is an unbounded left Hilbert algebra over $\lambda_f(\mathbf{A}_b(f))$ in $\mathfrak{H}_f$.

LEMMA 3.1. *If there exists a *-subalgebra $\mathbf{B}(f)$ of $\mathbf{A}_b(f)$ with another involution $a \rightarrow a^\flat$ such that*

(i) *$\lambda_f(\mathbf{B}(f))$ is dense in $\mathfrak{H}_f$;*

(ii) *$f(a^*b) = f(ba^\flat)$ for all $a \in \mathbf{B}(f)$ and $a \in \mathbf{A}$, then*

(1) *$\mathbf{N}_f$ is a *-ideal in $\mathbf{A}$;*

(2) *$\lambda_f(\mathbf{A}_b(f))$ is left Hilbert algebra in $\mathfrak{H}_f$ under the operations: $\lambda_f(a)\,\lambda_f(b) \equiv \pi_0(\lambda_f(a))\,\lambda_f(b) = \lambda_f(ab)$, $\lambda_f(a)^\# = \lambda_f(a^*)$;*

(3) *$\lambda_f(\mathbf{B}(f)) \subset \mathscr{D}^\flat(\lambda_f(\mathbf{A}_b(f)))$ and $\lambda_f(b)^\flat = \lambda_f(b^\flat)$ for every $b \in \mathbf{B}(f)$;*

(4) *$\lambda_f(e)$ is a generating and separating vector for the von Neumann algebra $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$ and*

$$\lambda_f(\mathbf{A}_b(f))' = \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))\,\lambda_f(e), \qquad \lambda_f(\mathbf{A}_b(f))'' = \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))\,\lambda_f(e).$$

Proof. (1) It is obvious that $\mathbf{N}_f$ is a left ideal in $\mathbf{A}$. Suppose that $a \in \mathbf{N}_f$. Then, for each $b \in \mathbf{B}(f)$ we have

$$\begin{aligned}(\lambda_f(b) \mid \lambda_f(a^*)) &= f(ab) = f((b^\flat)^* a)\\ &= (\lambda_f(a) \mid \lambda_f(b^\flat)) = 0.\end{aligned}$$

From the density of $\lambda_f(\mathbf{B}(f))$ in $\mathfrak{H}_f$, it follows that $\lambda_f(a^*) = 0$, i.e., $a^* \in \mathbf{N}_f$. Thus, $\mathbf{N}_f$ is a *-ideal in $\mathbf{A}$.

(2) By (1), we can easily show that $\lambda_f(\mathbf{A}_b(f))$ is an involutive algebra with the involution $\#$. It is obvious that $(\lambda_f(a)\,\lambda_f(b) \mid \lambda_f(c)) = (\lambda_f(b) \mid \lambda_f(a)^\#\,\lambda_f(c))$ for every $a, b, c \in \mathbf{A}_b(f)$ and $\lambda_f(\mathbf{A}_b(f))^2$ is dense in $\mathfrak{H}_f$. Further, from the equality: for each $b \in \mathbf{B}(f)$ and $a \in \mathbf{A}_b(f)$

$$\begin{aligned}(\lambda_f(b^\flat) \mid \lambda_f(a)) &= f(a^*b^\flat)\\ &= f(b^*a^*)\\ &= (\lambda_f(a)^\# \mid \lambda_f(b))\end{aligned}$$

it follows that $\lambda_f(a) \rightarrow \lambda_f(a)^\#$ is closable as a real linear operator on the real pre-Hilbert space $\lambda_f(\mathbf{A}_b(f))$. Thus, $\lambda_f(\mathbf{A}_b(f))$ is a left Hilbert algebra in $\mathfrak{H}_f$.

(3) This follows from the proof of (2).

(4) It is obvious that $\lambda_f(e)$ is a generating vector for $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$. We show that $\lambda_f(e)$ is a separating vector for $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$. Suppose that $X\lambda_f(e) = 0$

for $X \in \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$. There exists a sequence $\{a_n\}$ in $\mathbf{A}_b(f)$ such that $\overline{\pi_f(a_n)}$ converges strongly to X. For each $b \in \mathbf{B}(f)$ we have

$$\begin{aligned}(\lambda_f(b) \mid X^*\lambda_f(e)) &= \lim_{n\to\infty} (\pi_f(a_n)\,\lambda_f(b) \mid \lambda_f(e)) \\ &= \lim_{n\to\infty} f(a_n b) \\ &= \lim_{n\to\infty} f((b^\flat)^*\, a_n) \\ &= \lim_{n\to\infty} (\lambda_f(a_n) \mid \lambda_f(b^\flat)) \\ &= \lim_{n\to\infty} (\pi_f(a_n)\,\lambda_f(e) \mid \lambda_f(b^\flat)) \\ &= (X\lambda_f(e) \mid \lambda_f(b^\flat)) \\ &= 0.\end{aligned}$$

Hence, $X^*\lambda_f(e) = 0$. Since

$$\begin{aligned}(X\lambda_f(b) \mid \lambda_f(c)) &= \lim_{n\to\infty} (\pi_f(a_n)\,\lambda_f(b) \mid \lambda_f(c)) \\ &= \lim_{n\to\infty} f(c^* a_n b) \\ &= \lim_{n\to\infty} f(a_n b c^\flat) \\ &= \lim_{n\to\infty} (\pi_f(a_n)\,\lambda_f(bc^\flat) \mid \lambda_f(e)) \\ &= (X\lambda_f(bc^\flat) \mid \lambda_f(e)) \\ &= (\lambda_f(bc^\flat) \mid X^*\lambda_f(e)) \\ &= 0\end{aligned}$$

for each $b, c \in \mathbf{B}(f)$ and $\lambda_f(\mathbf{B}(f))$ is dense in $\mathfrak{H}_f$, we have $X = 0$. Thus, $\lambda_f(e)$ is a separating vector for $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$. From Theorem 12.1 in [10], it follows that

$$\lambda_f(\mathbf{A}_b(f))' = \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))\,\lambda_f(e) \qquad \text{and} \qquad \lambda_f(\mathbf{A}_b(f))'' = \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))\,\lambda_f(e).$$

We consider the converse of Lemma 3.1.

LEMMA 3.2. *If f is a faithful positive linear functional such that $\lambda_f(\mathbf{A}_b(f))$ is an achieved left Hilbert algebra in $\mathfrak{H}_f$, then there exists a *-subalgebra $\mathbf{B}(f)$ of $\mathbf{A}_b(f)$ with an involution $a \to a^\flat$ such that $\lambda_f(\mathbf{B}(f))$ is dense in $\mathfrak{H}_f$ and $f(a^*b) = f(ba^\flat)$ for all $a \in \mathbf{B}(f)$ and $b \in \mathbf{A}_b(f)$.*

Proof. Let $\mathscr{B}_0(f)$ be a Tomita algebra equivalent to the left Hilbert algebra $\lambda_f(\mathbf{A}_b(f))$. We set

$$\mathbf{B}(f) = \{a \in \mathbf{A}_b(f);\, \lambda_f(a) \in \mathscr{B}_0(f)\}.$$

Then, it is easily proved that $\mathbf{B}(f)$ is a $*$-subalgebra of $\mathbf{A}_b(f)$ and an involution $a \to a^\flat$ on $\mathbf{B}(f)$ is defined by: $\lambda_f(a^\flat) = \lambda_f(a)^\flat$. Further, since $\lambda_f(\mathbf{A}_b(f))$ is achieved, $\lambda_f(\mathbf{B}(f)) = \mathscr{B}_0(f)$. And so, $\lambda_f(\mathbf{B}(f))$ is dense in $\mathfrak{H}_f$. For each $a \in \mathbf{B}(f)$ and $b \in A_b(f)$ we have

$$\begin{aligned} f(a^*b) &= (\lambda_f(b) \mid \lambda_f(a)) \\ &= (\lambda_f(a)^\flat \mid \lambda_f(b)^\#) \\ &= (\lambda_f(a^\flat) \mid \lambda_f(b^*)) \\ &= f(ba^\flat). \end{aligned}$$

DEFINITION 3.1. If f satisfies conditions (i) and (ii) of Lemma 3.1 and

(iii) $\lambda_f(\mathbf{B}(f))$ is densely contained in $\lambda_f(\mathbf{A}_b(f))'$ with respect to the norm topology in the Hilbert space $\mathscr{D}^\flat(\lambda_f(\mathbf{A}_b(f)))$, then f is called quasi-abelian.

THEOREM 3.1. *If f is quasi-abelian, then $\lambda_f(\mathbf{A})$ is an unbounded left Hilbert algebra over $\lambda_f(\mathbf{A}_b(f))$ in $\mathfrak{H}_f$ under the operations*: $\lambda_f(a)\,\lambda_f(b) \equiv \pi(\lambda_f(a))\,\lambda_f(b) = \lambda_f(ab)$, $\lambda_f(a)^\# = \lambda_f(a^*)$.

Proof. It follows from Lemma 3.1 (1) that $\lambda_f(\mathbf{A})$ is a $*$-algebra. Further, it is easily proved that $(\lambda_f(a)\,\lambda_f(b) \mid \lambda_f(c)) = (\lambda_f(b) \mid \lambda_f(a)^\#\,\lambda_f(c))$ for every $a, b, c \in \mathbf{A}$ and $\lambda_f(\mathbf{A})_0 = \{\lambda_f(a) \in \lambda_f(\mathbf{A}); \overline{\pi(\lambda_f(a))} \in \mathscr{B}(\mathfrak{H}_f)\} = \lambda_f(\mathbf{A}_b(f))$. Hence, from Lemma 3.1 (2) it follows that $\lambda_f(\mathbf{A})_0$ is a left Hilbert algebra in $\mathfrak{H}_f$.

We next show that $\lambda_f(\mathbf{A}) \subset \mathscr{D}^\#(\lambda_f(\mathbf{A}_b(f)))$ and $S_{\lambda_f(\mathbf{A}_b(f))}\lambda_f(a) = \lambda_f(a)^\#$ for every $a \in \mathbf{A}$. It follows from assumption (iii) that for each $X' \in \overline{\pi_f(\mathbf{A}_b(f))}'$ there exists a sequence $\{b_n\}$ in $\mathbf{B}(f)$ such that $\overline{\pi_0(\lambda_f(b_n))}$ converges strongly to X'. For each $a \in \mathbf{A}$ we have

$$\begin{aligned} (\lambda_f(a) \mid X'\lambda_f(e)) &= \lim_{n\to\infty} (\lambda_f(a) \mid \pi_0'(\lambda_f(b_n))\,\lambda_f(e)) \\ &= \lim_{n\to\infty} (\lambda_f(a) \mid \lambda_f(b_n)) \\ &= \lim_{n\to\infty} f(b_n^*a) \\ &= \lim_{n\to\infty} f(ab_n{}^\flat) \\ &= \lim_{n\to\infty} (\lambda_f(b_n{}^\flat) \mid \lambda_f(a)^\#) \\ &= \lim_{n\to\infty} (\pi_0'(\lambda_f(b_n))^*\,\lambda_f(e) \mid \lambda_f(a)^\#) \\ &= ((X')^*\,\lambda_f(e) \mid \lambda_f(a)^\#) \\ &= ((X'\lambda_f(e))^\flat \mid \lambda_f(a)^\#). \end{aligned}$$

This implies that $\lambda_f(\mathbf{A}) \subset \mathscr{D}^\#(\lambda_f(\mathbf{A}_b(f)))$ and $S_{\lambda_f(\mathbf{A}_b(f))}\lambda_f(a) = \lambda_f(a)^\#$ for every $a \in \mathbf{A}$.

We now show that $\pi_f(a)\,\lambda_f(b) = \overline{\pi_0'(\lambda_f(b))}\,\lambda_f(a)$ for every $a \in \mathbf{A}$ and $b \in \mathbf{B}(f)$. This follows from the equalities: for each $c \in \mathbf{B}(f)$

$$\begin{aligned}(\pi_f(a)\,\lambda_f(b) \mid \lambda_f(c)) &= f(c^*ab)\\ &= f((b^\flat)^*\,c^*a)\end{aligned}$$

and

$$\begin{aligned}(\overline{\pi_0'(\lambda_f(b))}\,\lambda_f(a) \mid \lambda_f(c)) &= (\lambda_f(a) \mid \pi_0'(\lambda_f(b))^*\,\lambda_f(c))\\ &= (\lambda_f(a) \mid \lambda_f(cb^\flat))\\ &= f((b^\flat)^*\,c^*a).\end{aligned}$$

Using this fact, we can prove that $\overline{\pi_f(a)}\,\eta\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$ for every $a \in \mathbf{A}$. In fact, suppose that $X' \in \mathscr{V}_0(\lambda_f(A_b(f)))$. Since $X'\overline{\pi_f(a)} \subset \overline{X'\pi_f(a)}$, we need only show that $\overline{X'\pi_f(a)} \subset \overline{\pi_f(a)}\,X'$. By assumption (iii) there exists a sequence $\{c_n\}$ in $\mathbf{B}(f)$ such that $\overline{\pi_0'(\lambda_f(c_n))}$ converges strongly to X'. Then, for each $b \in \mathbf{A}$ we have

$$\begin{aligned}\lim_{n\to\infty} \lambda_f(bc_n) &= \lim_{n\to\infty} \pi_0'(\lambda_f(c_n))\,\lambda_f(b)\\ &= X'\lambda_f(b)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \overline{\pi_f(a)}\,\lambda_f(bc_n) &= \lim_{n\to\infty} \pi_f(ab)\,\lambda_f(c_n)\\ &= \lim_{n\to\infty} \overline{\pi_0'(\lambda_f(c_n))}\,\lambda_f(ab)\\ &= X'\lambda_f(ab).\end{aligned}$$

Hence, $X'\lambda_f(b) \in \mathscr{D}(\overline{\pi_f(a)})$ and $\overline{\pi_f(a)}\,X'\lambda_f(b) = X'\pi_f(a)\,\lambda_f(b)$. Thus, $\overline{X\pi_f(a)} \subset \overline{\pi_f(a)}\,X'$.

Finally, we show that $\overline{\pi_0(\lambda_f(a))} \subset \overline{\pi(\lambda_f(a))}$ for every $a \in \mathbf{A}$. This follows from the equality

$$\begin{aligned}\pi_0(\lambda_f(a))\,X'\lambda_f(e) &= X'\lambda_f(a)\\ &= X'\pi_f(a)\,\lambda_f(e)\\ &= \overline{\pi_f(a)}\,X'\lambda_f(e)\\ &= \overline{\pi(\lambda_f(a))}\,X'\lambda_f(e)\end{aligned}$$

for every $X' \in \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$.

This completes the proof.

We consider the converse of Theorem 3.1.

THEOREM 3.2. *If f is faithful and $\lambda_f(\mathbf{A})$ is an achieved unbounded left Hilbert algebra over $\lambda_f(\mathbf{A}_b(f))$ in $\mathfrak{H}_f$, then f is quasi-abelian.*

Proof. This is proved in the same way as in Lemma 3.2.

COROLLARY 3.1. *Let $\mathbf{A}$ be a topological *-algebra with identity e and f a continuous faithful positive linear functional on $\mathbf{A}$. If $\mathbf{A}_b(f)$ is dense in $\mathbf{A}$ and $\lambda_f(\mathbf{A}_b(f))$ is an achieved left Hilbert algebra in $\mathfrak{H}_f$, then f is quasi-abelian and $\lambda_f(\mathbf{A})$ is an achieved unbounded left Hilbert algebra over $\lambda_f(\mathbf{A}_b(f))$ in $\mathfrak{H}_f$.*

Proof. From Lemma 3.2, there exists a *-subalgebra $\mathbf{B}(f)$ of $\mathbf{A}_b(f)$ with an involution $a \to a^\flat$ such that $\lambda_f(\mathbf{B}(f))$ is a Tomita algebra equivalent to the left Hilbert algebra $\lambda_f(\mathbf{A}_b(f))$ and $f(a^*b) = f(ba^\flat)$ for all $a \in \mathbf{B}(f)$ and $b \in \mathbf{A}_b(f)$. Since f is continuous and $\mathbf{A}_b(f)$ is dense in $\mathbf{A}$, we have

$$f(a^*b) = f(ba^\flat)$$

for all $a \in \mathbf{B}(f)$ and $b \in \mathbf{A}$. Thus, we can see that f is quasi-abelian. From Theorem 3.1, $\lambda_f(\mathbf{A})$ is an unbounded left Hilbert algebra over $\lambda_f(\mathbf{A}_b(f))$ in $\mathfrak{H}_f$.

COROLLARY 3.2. *If $\mathscr{M}$ is a closed $EW^\#$-algebra over $\mathscr{M}_0$ and ϕ is a σ-weakly continuous (cf. [3]) faithful positive linear functional on $\mathscr{M}$, then λ_ϕ is a faithful *-representation of $\mathscr{M}$ onto the unbounded left Hilbert algebra $\lambda_\phi(\mathscr{M})$ over $\lambda_\phi(\mathscr{M}_0)$ in $\mathfrak{H}_\phi$.*

Proof. From Theorem 4.2 in [8], $\pi_\phi(\mathscr{M})$ is an $EW^\#$-algebra over $\pi_\phi(\mathscr{M}_0)$ and $\lambda_\phi(I)$ is a generating and separating vector for the von Neumann algebra $\pi_\phi(\mathscr{M}_0)$ (where I denotes the identity operator). Further, we can show that $\mathscr{M}_0 = \mathscr{M}_b(f) = \{A \in \mathscr{M}; \overline{\pi_\phi(A)} \in \mathscr{B}(\mathfrak{H}_\phi)\}$ and $\mathscr{M}_0$ is σ-weakly dense in $\mathscr{M}$. Hence, the corollary follows from Corollary 3.1.

Below we give the other condition under which $\lambda_f(A)$ is an unbounded left Hilbert algebra.

THEOREM 3.3. *If there exists a *-subalgebra $\mathbf{B}(f)$ of $\mathbf{A}_b(f)$ with an involution $a \to a^\flat$ such that*

(i) *$\lambda_f(\mathbf{B}(f))$ is dense in $\mathfrak{H}_f$;*

(ii) *$f(a^*b) = f(ba^\flat)$ for all $a \in \mathbf{B}(f)$ and $b \in \mathbf{A}$;*

(iii) *$\overline{\pi_f(a)}\, \eta\, \overline{\pi_f(\mathbf{A}_b(f))''}$ for every $a \in \mathbf{A}$, then $\lambda_f(\mathbf{A})$ is an unbounded left Hilbert algebra over $\lambda_f(\mathbf{A}_b(f))$ in $\mathfrak{H}_f$.*

Proof. From Lemma 3.1, $\lambda_f(\mathbf{A}_b(f))$ is a left Hilbert algebra in $\mathfrak{H}_f$ and $\lambda_f(e)$ is a generating and separating vector for $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$. Further, we have that $\lambda_f(\mathbf{A}_b(f))' = \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))\,\lambda_f(e)$ and $\lambda_f(\mathbf{A}_b(f))'' = \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))\,\lambda_f(e)$. We need only prove that

(1) $\lambda_f(\mathbf{A}) \subset \mathscr{D}^\#(\lambda_f(\mathbf{A}_b(f)))$ and $S_{\lambda_f(\mathbf{A}_b(f))}\lambda_f(a) = \lambda_f(a)^\#$ for every $a \in \mathbf{A}$;

(2) $\overline{\pi(\lambda_f(a))} \supset \overline{\pi_0(\lambda_f(a))}$ for every $a \in \mathbf{A}$.

(1) For each $a \in \mathbf{A}$ let $\overline{\pi_f(a)} = U \mid \overline{\pi_f(a)}|$ be the polar decomposition of $\overline{\pi_f(a)}$ and let $| \overline{\pi_f(a)}| = \int_0^\infty \lambda \, dE(\lambda)$ be the spectral resolution of $| \overline{\pi_f(a)}|$. We set

$$A_n = \int_0^n \lambda \, dE(\lambda), \qquad n = 1, 2, \ldots .$$

By assumption (iii), it follows that $U, A_n \in \overline{\pi_f(\mathbf{A}_b(f))}'' = \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$. Then, we have that $UA_n\lambda_f(e) \in \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))\, \lambda_f(e) = \lambda_f(\mathbf{A}_b(f))''$ and

$$\lim_{n\to\infty} UA_n\lambda_f(e) = \lambda_f(a)$$

$$\begin{aligned}\lim_{n\to\infty} (UA_n\lambda_f(e))^\# &= \lim_{n\to\infty} A_n^* U^* \lambda_f(e) \\ &= \lambda_f(a^*) \\ &= \lambda_f(a)^\#.\end{aligned}$$

Hence, it follows that $\lambda_f(a) \in \mathscr{D}^\#(\lambda_f(\mathbf{A}_b(f)))$ and $S_{\lambda_f(\mathbf{A}_b(f))}\lambda_f(a) = \lambda_f(a)^\#$.

(2) This follows immediately from $\overline{\pi_f(a)}\, \eta \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$ and $\lambda_f(\mathbf{A}_b(f))' = \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))\, \lambda_f(e)$.

This completes the proof.

COROLLARY 3.3. *Let* $\mathbf{A}$ *be a symmetric *-algebra* (*that is*, $(e + x^*x)^{-1}$ *exists in* $\mathbf{A}$ *for every* $x \in \mathbf{A}$). *If* f *satisfies conditions* (i) *and* (ii) *of Theorem* 3.3, *then* $\lambda_f(\mathbf{A})$ *is an unbounded left Hilbert algebra over* $\lambda_f(\mathbf{A}_b(f))$ *in* $\mathfrak{H}_f$.

Proof. From Lemma 3.2 in [4] it follows that $\pi_f(\mathbf{A})$ is a symmetric $^\#$-algebra on $\lambda_f(\mathbf{A})$ (cf. [3]). Further, from Lemmas 1 and 2 in [9] we have that $\overline{\pi_f(a)}\, \eta \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$ for every $a \in \mathbf{A}$. Hence, the corollary follows from Theorem 3.3.

4. UNBOUNDED-REPRESENTABLE POSITIVE LINEAR FUNCTIONALS

Let $\mathbf{A}$ be a *-algebra with identity e and f a quasi-abelian positive linear functional on $\mathbf{A}$.

In this section we study positive linear functionals g with $g \leqslant f$.

DEFINITION 4.1. A positive linear functional g on $\mathbf{A}$ is called f-quasiabelian if $g(a^*b) = g(ba^\flat)$ for all $a \in \mathbf{B}(f)$ and $a \in \mathbf{A}$.

For $T \in \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ with $T \geqslant 0$ we set

$$f_T(x) = (T\lambda_f(x) \mid \lambda_f(e)), \qquad x \in \mathbf{A}.$$

Then, since $\overline{\pi_f(x)}\, \eta \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$, f_T is a positive linear functional on $\mathbf{A}$ with $f_T \leqslant \| T \| f$.

LEMMA 4.1. *If g is a positive linear functional on $\mathbf{A}$ with $g \leqslant f$, then the following conditions are equivalent*:

(1) *g is f-quasiabelian*;

(2) *there exists an element T of $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f))) \cap \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ such that $0 \leqslant T \leqslant I$ and $g = f_T$.*

Proof. (1) $\Rightarrow$ (2) It is immediately proved that there exists an element T of $\mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ such that $0 \leqslant T \leqslant I$ and $g = f_T$. Hence, we have only to show that $T \in \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$. For each $a, b, c \in \mathbf{B}(f)$ we have

$$\begin{aligned}
(\overline{\pi_0'(\lambda_f(a))}\, T\lambda_f(c) \mid \lambda_f(b)) &= (T\lambda_f(c) \mid \pi_0'(\lambda_f(a))^* \lambda_f(b)) \\
&= (T\lambda_f(c) \mid \lambda_f(b)\, \lambda_f(a^\flat)) \\
&= (T\lambda_f((a^\flat)^* b^* c) \mid \lambda_f(e)) \\
&= g((a^\flat)^* b^* c) \\
&= g(b^* ca) \\
&= (T\lambda_f(b^* ca) \mid \lambda_f(e)) \\
&= (T\pi_0'(\lambda_f(a))\, \lambda_f(c) \mid \lambda_f(b)).
\end{aligned}$$

Hence, it follows that $\overline{\pi_0'(\lambda_f(a))}\, T = T\overline{\pi_0'(\lambda_f(a))}$ for every $a \in \mathbf{B}(f)$. Since $\overline{\pi_0'(\lambda_f(\mathbf{B}(f)))}'' = \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$, we have $T \in \mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)))$.

(2) $\Rightarrow$ (1) For each $a \in \mathbf{B}(f)$ and $b \in \mathbf{A}$ we have

$$\begin{aligned}
g(a^* b) &= (T\lambda_f(a^* b) \mid \lambda_f(e)) \\
&= (T\lambda_f(b) \mid \lambda_f(a))
\end{aligned}$$

and

$$\begin{aligned}
g(ba^\flat) &= (T\lambda_f(ba^\flat) \mid \lambda_f(e)) \\
&= (T\overline{\pi_0'(\lambda_f(a^\flat))}\, \lambda_f(b) \mid \lambda_f(e)) \\
&= (\pi_0'(\lambda_f(a))^*\, T\lambda_f(b) \mid \lambda_f(e)) \\
&= (T\lambda_f(b) \mid \lambda_f(a)).
\end{aligned}$$

Hence, g is f-quasiabelian.

LEMMA 4.2. *If E is a projection in $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f))) \cap \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$, then f_E is a quasi-abelian positive linear functional on $\mathbf{A}$ with $f_E \leqslant f$. Further, $\lambda_{f_E}(A)$ is an unbounded left Hilbert algebra which is identified with the unbounded left Hilbert algebra $E\lambda_f(\mathbf{A})$.*

Proof. We set

$$\mathbf{B}(f_E) = \mathbf{B}(f).$$

For each $x \in \mathbf{A}_b(f)$ and $y \in \mathbf{A}$ we have

$$f_E(y^*x^*xy) \leqslant \| \overline{\pi_f(x)} \|^2 f_E(y^*y).$$

Hence, $\mathbf{A}_b(f) \subset \mathbf{A}_b(f_E)$. Thus, $\mathbf{B}(f_E)$ is a *-subalgebra of $\mathbf{A}_b(f_E)$ with an involution $a \to a^\flat$. Also, it is easily proved that $\lambda_{f_E}(\mathbf{B}(f_E))$ is dense in $\mathfrak{H}_f$. From Lemma 4.1, f_E is f-quasi-abelian. Hence, it follows from Lemma 3.1 that $\mathbf{N}_{f_E}$ is a *-ideal in $\mathbf{A}$ and $\lambda_{f_E}(\mathbf{A}_b(f_E))$ is a left Hilbert algebra in $\mathfrak{H}_{f_E}$. Also, $\lambda_{f_E}(\mathbf{A})$ is a pre-Hilbert space in $\mathfrak{H}_{f_E}$ and an involutive algebra with involution $\#$. By the map U: $\lambda_{f_E}(x) \to E\lambda_f(x)$, it is easily proved that U is extended to the isometric map from $\mathfrak{H}_{f_E}$ onto $E\mathfrak{H}_f$ and $U(\lambda_{f_E}(x)\,\lambda_{f_E}(y)) = (U\lambda_{f_E}(x))(U\lambda_{f_E}(y))$, $U\lambda_{f_E}(x)^\# = (U\lambda_{f_E}(x))^\#$. Hence, it follows that $\lambda_{f_E}(\mathbf{A})$ is identified with the unbounded left Hilbert algebra $E\lambda_f(\mathbf{A})$. By identifying $\lambda_{f_E}(\mathbf{B}(f_E))$ (resp. $\lambda_{f_E}(\mathbf{A}_b(f_E))'$) with $E\lambda_f(\mathbf{B}(f))$ (resp. $(E\lambda_f(\mathbf{A}_b(f_E)))' = E\lambda_f(\mathbf{A}_b(f))'$), we can show that $\lambda_{f_E}(\mathbf{B}(f_E))$ is densely contained in $\lambda_{f_E}(\mathbf{A}_b(f_E))'$ with respect to the norm topology in the Hilbert space $\mathscr{D}^\flat(\lambda_{f_E}(\mathbf{A}_b(f_E)))$. Thus, f_E is quasi-abelian.

DEFINITION 4.2. Let g be a positive linear functional on $\mathbf{A}$. If $\lambda_g(\mathbf{A}_b(g))$ is a left Hilbert algebra in $\mathfrak{H}_g$ under the operations: $\lambda_g(a)\,\lambda_g(b) \equiv \pi_0(\lambda_g(a))\,\lambda_g(b) = \lambda_g(ab)$, $\lambda_g(a)^\# \equiv \lambda_g(a^*)$, then g is called bounded-representable. If $\lambda_g(\mathbf{A})$ is an unbounded left Hilbert algebra over $\lambda_g(\mathbf{A}_b(g))$ in $\mathfrak{H}_g$ under the operations: $\lambda_g(a)\,\lambda_g(b) \equiv \pi(\lambda_g(a))\,\lambda_g(b) = \lambda_g(ab)$, $\lambda_g(a)^\# \equiv \lambda_g(a^*)$, then g is called unbounded-representable.

LEMMA 4.3. *If g is an f-quasiabelian positive linear functional on $\mathbf{A}$ with $g \leqslant f$, then g is bounded-representable. Further, $\lambda_g(e)$ is a generating and separating vector for the von Neumann algebra $\mathscr{U}_0(\lambda_g(\mathbf{A}_b(g))$ and*

$$\lambda_g(\mathbf{A}_b(g))' = \mathscr{V}_0(\lambda_g(\mathbf{A}_b(g)))\,\lambda_g(e), \qquad \lambda_g(\mathbf{A}_b(g))'' = \mathscr{U}_0(\lambda_g(\mathbf{A}_b(g)))\,\lambda_g(e).$$

Proof. By Lemma 4.1, there exists an element T of $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f)) \cap \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ such that $0 \leqslant T \leqslant I$ and $g = f_T$. For each $a \in \mathbf{A}_b(f)$ and $b \in \mathbf{A}$ we have

$$\begin{aligned} \| \pi_g(a)\,\lambda_g(b) \|^2 &= \| \overline{\pi_f(a)}\, T^{1/2}\lambda_f(b) \|^2 \\ &\leqslant \| \overline{\pi_f(a)} \|^2 \, \| T^{1/2}\lambda_f(b) \|^2 \\ &= \| \overline{\pi_f(a)} \|^2 \, \| \lambda_g(b) \|^2. \end{aligned}$$

Hence, it follows that $\mathbf{A}_b(f) \subset \mathbf{A}_b(g)$. For each $x \in \mathbf{A}$ and $b \in \mathbf{B}(f)$ we have

$$\begin{aligned}(T^{1/2}\lambda_f(x^*) \mid \lambda_f(b)) &= (T^{1/2}\lambda_f(e) \mid \pi_f(x)\,\lambda_f(b)) \\ &= (T^{1/2}\lambda_f(e) \mid \overline{\pi_0'(\lambda_f(b))}\,\lambda_f(x)) \\ &= (\lambda_f(b^\flat) \mid T^{1/2}\lambda_f(x)).\end{aligned}$$

Hence, it follows that the left ideal $\mathbf{N}_g = \{x \in \mathbf{A};\, g(x^*x) = 0\}$ is a $*$-ideal in $\mathbf{A}$. And so, $\lambda_g(\mathbf{A}_b(g))$ is an involutive algebra with the involution $\#$. For each $a \in \mathbf{A}$ there exists a sequence $\{b_n\}$ in $\mathbf{B}(f)$ such that $\lim_{n\to\infty} \lambda_f(b_n) = \lambda_f(a)$. Then, we have

$$\begin{aligned}\lim_{n\to\infty} \| \lambda_g(b_n) - \lambda_g(a)\|^2 &= \lim_{n\to\infty} (T\lambda_f(b_n - a) \mid \lambda_f(b_n - a)) \\ &= 0.\end{aligned}$$

Hence, $\lambda_g(\mathbf{B}(f))$ is dense in $\mathfrak{H}_g$. Also, since $\lambda_g(\mathbf{B}(f)) \subset \lambda_g(\mathbf{A}_b(f)) \subset \lambda_g(\mathbf{A}_b(g))$, $\lambda_g(\mathbf{A}_b(g))$ is dense in $\mathfrak{H}_g$. Further, by the equality: for each $a \in \mathbf{A}_b(g)$ and $b \in \mathbf{B}(f)$

$$(\lambda_g(a) \mid \lambda_g(b)) = g(b^*a) = g(ab^\flat) = (\lambda_g(b^\flat) \mid \lambda_g(a)^\#)$$

it follows that $\lambda_f(a) \to \lambda_f(a)^\#$ is closable.

Thus, we can see that $\lambda_g(\mathbf{A}_b(g))$ is a left Hilbert algebra in $\mathfrak{H}_g$.

We can prove in the same way as in Lemma 3.1 (4) that $\lambda_g(e)$ is a generating and separating vector for $\mathscr{U}_0(\lambda_g(\mathbf{A}_b(g)))$ and $\lambda_g(\mathbf{A}_b(g))' = \mathscr{V}_0(\lambda_g(\mathbf{A}_b(g)))\,\lambda_g(e)$, $\lambda_g(\mathbf{A}_b(g))'' = \mathscr{U}_0(\lambda_g(\mathbf{A}_b(g)))\,\lambda_g(e)$.

We consider below the conditions under which g is unbounded-representable.

LEMMA 4.4. *If g is a quasi-abelian positive linear functional on $\mathbf{A}$, then it is unbounded-representable.*

LEMMA 4.5. *Let A be a symmetric $*$-algebra. If g is f-quasiabelian, then g is unbounded-representable.*

Proof. This follows from Corollary 3.3 and Lemma 4.3.

LEMMA 4.6. *If g is an unbounded-representable positive linear functional on $\mathbf{A}$ and $\mathbf{E}$ is a projection in $\mathscr{U}_0(\lambda_g(\mathbf{A}_b(g))) \cap \mathscr{V}_0(\lambda_g(\mathbf{A}_b(g)))$, then g_E is unbounded-representable. And, the unbounded left Hilbert algebra $\lambda_{g_E}(\mathbf{A})$ is identified with the unbounded left Hilbert algebra $E\lambda_g(\mathbf{A})$.*

Proof. This is proved in the same way as in Lemma 4.2.

THEOREM 4.1. *Let* $\mathbf{A}$ *be a topological* $*$*-algebra with identity* e *and* f *a continuous quasi-abelian positive linear functional on* $\mathbf{A}$ *such that* $\mathbf{A}_b(f)$ *is dense in* $\mathbf{A}$. *If* g *is a continuous* f*-quasiabelian positive linear functional on* $\mathbf{A}$ *with* $g \leqslant f$ *such that there exists a Tomita algebra* $\mathscr{T}(g)$ *which is equivalently contained in the left Hilbert algebra* $\lambda_g(\mathbf{A}_b(g))$, *then* g *is unbounded-representable.*

Proof. By Lemma 4.3, $\lambda_g(\mathbf{A}_b(g))$ is a left Hilbert algebra in $\mathfrak{H}_g$. Also, it follows from $\mathbf{A}_b(f) \subset \mathbf{A}_b(g)$ that $\mathbf{A}_b(g)$ is dense in $\mathbf{A}$. From the assumption: $\mathscr{T}(g) \subset \lambda_g(\mathbf{A}_b(g))$ it follows that for each $\eta \in \mathscr{T}(g)$ there exist elements $b, c \in \mathbf{A}_b(g)$ such that $\eta = \lambda_g(b)$ *and* $\eta^\flat = \lambda_g(c)$. Then, for each $a \in \mathbf{A}_b(g)$, we have

$$\begin{aligned} g(b^*a) &= (\lambda_g(a) \mid \eta) = (\eta^\flat \mid \lambda_g(a)^\#) \\ &= (\lambda_g(c) \mid \lambda_g(a^*)) = g(ac). \end{aligned}$$

Since $\mathbf{A}_b(g)$ is dense in $\mathbf{A}$ and g is continuous, we have

$$(\lambda_g(a) \mid \eta) = g(b^*a) = g(ac) = (\eta^\flat \mid \lambda_g(a)^\#)$$

for every $a \in \mathbf{A}$. Thus, $\lambda_g(a) \in \mathscr{D}^\#(\lambda_g(\mathbf{A}_b(g)))$ and $S_{\lambda_g(\mathbf{A}_b(g))}\lambda_g(a) = \lambda_g(a)^\#$.

We now show that $\pi_g(a)\,\eta = \overline{\pi_0'(\eta)}\,\lambda_g(a)$ for each $a \in \mathbf{A}$ and $\eta \in \mathscr{T}(g)$, where π_0 (resp. π_0') denotes the left (resp. right) regular representation of the left Hilbert algebra $\lambda_g(\mathbf{A}_b(g))$. This follows from the equality: for each $a, x \in \mathbf{A}_b(g)$

$$\begin{aligned} g(x^*ab) &= (\pi_g(a)\eta \mid \lambda_g(x)) \qquad (\eta \equiv \lambda_g(b),\, b \in \mathbf{A}_b(g)) \\ &= (\overline{\pi_0'(\eta)}\,\lambda_g(a) \mid \lambda_g(x)) \\ &= (\lambda_g(a) \mid \pi_0'(\eta^\flat)\,\lambda_g(x)) \\ &= (\lambda_g(a) \mid \lambda_g(xc)) \qquad (\eta^\flat \equiv \lambda_g(c),\, c \in \mathbf{A}_b(g)) \\ &= g(c^*x^*a), \end{aligned}$$

the density of $\mathbf{A}_b(g)$ in $\mathbf{A}$ and the continuity of g. Using the above fact, we can prove in the same way as the proof of Theorem 3.1 that $\overline{\pi_g(a)}\,\eta\mathscr{U}_0(\lambda_g(\mathbf{A}_b(g)))$ for every $a \in \mathbf{A}$.

We finally claim that $\overline{\pi_0(\lambda_g(a))} \subset \overline{\pi(\lambda_g(a))}$ for every $a \in \mathbf{A}$. This follows from the equality: for each $X' \in \mathscr{V}_0(\lambda_g(\mathbf{A}_b(g)))$

$$\begin{aligned} \pi_0(\lambda_g(a))\,X'\lambda_g(e) &= X'\lambda_g(a) = X'\pi_g(a)\,\lambda_g(e) \\ &= \overline{\pi_g(a)}\,X'\lambda_g(e) = \overline{\pi(\lambda_g(a))}\,X'\lambda_g(e). \end{aligned}$$

Thus, we can see that $\lambda_g(\mathbf{A})$ is an unbounded left Hilbert algebra over $\lambda_g(\mathbf{A}_b(g))$ in $\mathfrak{H}_g$.

This completes the proof.

COROLLARY. *Suppose that* $\mathbf{A}$ *and* f *are of Theorem* 4.1. *If* g *is a continuous* f*-quasi-abelian positive linear functional on* $\mathbf{A}$ *with* $g \leqslant f$ *such that the left Hilbert algebra* $\lambda_g(\mathbf{A}_b(g))$ *is achieved, then* g *is unbounded-representable.*

Proof. This follows from Theorem 10.1 in [8] and theorem 4.1.

5. CLASSIFICATION OF QUASI-ABELIAN POSITIVE LINEAR FUNCTIONALS

Let $\mathbf{A}$ be a *-algebra with identity e and f a positive linear functional on $\mathbf{A}$.

DEFINITION 5.1. Let f be a quasi-abelian (resp. unbounded-representable) positive linear functional on $\mathbf{A}$. If $f(x^*a^*ax) \leqslant \gamma_a f(x^*x)$ (γ_a : constant) for all $a, x \in \mathbf{A}$, then f is called relatively bounded. If there exists a net $\{f_\alpha\}$ of quasi-abelian (resp. unbounded-representable) relatively bounded positive linear functionals on $\mathbf{A}$ with $f_\alpha \leqslant f$ which converges weakly to f, then f is called weakly relatively unbounded. If there does not exist any nonzero quasi-abelian (resp. unbounded-representable) relatively bounded positive linear functional g on $\mathbf{A}$ with $g \leqslant f$, then f is called strictly relatively unbounded.

THEOREM 5.1. *Let* f *be a quasi-abelian positive linear functional on* $\mathbf{A}$.

(i) f *is relatively bounded if and only if* $\lambda_f(\mathbf{A})$ *is a left Hilbert algebra.*

(ii) *The following conditions are equivalent*:

(1) f *is weakly relatively unbounded*;

(2) *there exists a net* $\{f_\alpha\}$ *of relatively bounded* f*-quasi-abelian positive linear functionals* f_α *with* $f_\alpha \leqslant f$ *which converges weakly to* f;

(3) *the unbounded left Hilbert algebra* $\lambda_f(\mathbf{A})$ *is weakly unbounded.*

(iii) *The following conditions are equivalent*:

(1) f *is strictly relatively unbounded*;

(2) *there does not exist any nonzero relatively bounded* f*-quasi-abelian positive linear functional* g *with* $g \leqslant f$;

(3) *the unbounded left Hilbert algebra* $\lambda_f(\mathbf{A})$ *is strictly unbounded.*

Proof. Assertion (i) is easily proved.

(ii) Assertion (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) From Lemma 4.1, there exists an element T_α of $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f))) \cap \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ such that $0 \leqslant T_\alpha \leqslant I$ and $f_\alpha = f_{T_\alpha}$. Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be a maximal family of nonzero mutually orthogonal projections in $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f))) \cap \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ such that f_{E_λ} is relatively bounded for every $\lambda \in \Lambda$. Then we

show that $\sum_{\lambda\in\Lambda} E_\lambda = I$. If $E \equiv \sum_{\lambda\in\Lambda} E_\lambda \neq I$, then there exists a nonzero element x of $\mathbf{A}$ such that $(I - E)\,\lambda_f(x) \neq 0$. Since $\{f_\alpha\}$ converges weakly to f, i.e., for each $y \in \mathbf{A}$

$$\begin{aligned}\lim_\alpha\,(T_\alpha\lambda_f(x) \mid \lambda_f(y)) &= \lim_\alpha f_\alpha(y^*x)\\ &= f(y^*x)\\ &= (\lambda_f(x) \mid \lambda_f(y))\end{aligned}$$

and $0 \leqslant T_\alpha \leqslant I$ for all α, $\{T_\alpha\lambda_f(x)\}$ converges weakly to $\lambda_f(x)$. And so, we have

$$\begin{aligned}\lim_\alpha\,(T_\alpha(I - E)\,\lambda_f(x) \mid \lambda_f(x)) &= \lim_\alpha\,(T_\alpha\lambda_f(x) \mid (I - E)\,\lambda_f(x))\\ &= (\lambda_f(x) \mid (I - E)\,\lambda_f(x))\\ &\neq 0.\end{aligned}$$

Hence, $T_{\alpha_0}(I - E)\lambda_f(x) \neq 0$ for some α_0. Let $T_{\alpha_0} = \int_0^1 \lambda\, dF_{\alpha_0}(\lambda)$ be the spectral resolution of T_{α_0}. Then, there exists a λ_0 such that $0 < \lambda_0 < 1$ and $(I - F_{\alpha_0}(\lambda_0))(I - E)\,\lambda_f(x) \neq 0$. We set

$$G_{\alpha_0} = (I - F_{\alpha_0}(\lambda_0))(I - E).$$

Since

$$f_{G_{\alpha_0}} \leqslant f_{I-F_{\alpha_0}(\lambda_0)} \leqslant \frac{1}{\lambda_0} f_{T_{\alpha_0}} = \frac{1}{\lambda_0} f_{\alpha_0}$$

and f_{α_0} is relatively bounded, we can prove that $f_{G_{\alpha_0}}$ is relatively bounded. This contradicts that $\{E_\lambda\}_{\lambda\in\Lambda}$ is maximal. By (i), $\lambda_{f_{E_\lambda}}(\mathbf{A})$ is a left Hilbert algebra for every $\lambda \in \Lambda$. Therefore, $E_\lambda\lambda_f(\mathbf{A})$ is a left Hilbert algebra for every $\lambda \in \Lambda$. Thus, $\lambda_f(\mathbf{A})$ is a weakly unbounded left Hilbert algebra.

(3) $\Rightarrow$ (1) Suppose that there exists a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of nonzero mutually orthogonal projections in $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f))) \cap \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ such that $\sum_{\lambda\in\Lambda} E_\lambda = I$ and $E_\lambda\lambda_f(\mathbf{A})$ is a left Hilbert algebra for every $\lambda \in \Lambda$. Then, it is easily proved that f_{E_λ} is relatively bounded for every $\lambda \in \Lambda$ and $\sum_{\lambda\in\Lambda} f_{E_\lambda}(x) = f(x)$ for every $x \in \mathbf{A}$. Hence, it follows from Lemma 4.2 that $\{f_\Delta\,;\, \Delta$ is a finite subset of $\Lambda\}$ is a net of relatively bounded quasi-abelian positive linear functionals on $\mathbf{A}$ with $f_\Delta \leqslant f$ which converges weakly to f. Thus, f is weakly relatively unbounded.

(iii) This is proved in the same way as (ii).

THEOREM 5.2. *Let f be an unbounded-representable positive linear functional on $\mathbf{A}$. Then, f is relatively bounded (resp. weakly relatively unbounded, strictly relatively unbounded) if and only if the unbounded left Hilbert algebra $\lambda_f(\mathbf{A})$ is a left Hilbert algebra (resp. weakly unbounded, strictly unbounded).*

Proof. Using Lemma 4.6, this is proved in the same way as in Theorem 5.1.

THEOREM 5.3. *If f is a quasi-abelian positive linear functional on $\mathbf{A}$, then there exist quasi-abelian positive linear functionals f_1, f_2 on $\mathbf{A}$ such that f_1 is weakly relatively unbounded, f_2 is strictly relatively unbounded, and $f = f_1 + f_2$.*

Proof. From Theorem 2.3 there exists a projection E in $\mathscr{U}_0(\lambda_f(\mathbf{A}_b(f))) \cap \mathscr{V}_0(\lambda_f(\mathbf{A}_b(f)))$ such that $E\lambda_f(\mathbf{A})$ is weakly unbounded and $(I - E)\,\lambda_f(\mathbf{A})$ is strictly unbounded. From Lemma 4.2 and Theorem 5.1, it is easily proved that $f_1 = f_E$ and $f_2 = f_{I-E}$ satisfy the assertions of the theorem.

COROLLARY. *If $\mathscr{M}$ is a closed $EW^{\#}$-algebra over $\mathscr{M}_0$ and ϕ is a faithful σ-weakly continuous positive linear functional on $\mathscr{M}$, then there exist quasi-abelian positive linear functionals ϕ_1, ϕ_2 on $\mathscr{M}$ such that ϕ_1 is weakly relatively unbounded, ϕ_2 is strictly relatively unbounded, and $\phi = \phi_1 + \phi_2$.*

Proof. This follows from Corollary 3.2 and Theorem 5.3.

THEOREM 5.4. *If f is an unbounded-representable positive linear functional on $\mathbf{A}$, then there exist unbounded-representable positive linear functionals f_1, f_2 on $\mathbf{A}$ such that f_1 is weakly relatively unbounded, f_2 is strictly relatively unbounded, and $f = f_1 + f_2$.*

Proof. This is proved in the same way as Theorem 5.3.

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